AIAA JOURNAL
Vol. 39, No. 2, February 2001

Maximal Errors due to Use of Equivalent Properties for Sublaminates

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Sublaminates are often used in design optimization to reduce the complexity of the design task, to simplify modeling, to reduce data input to analytical models, and to provide confidence in failure prediction. Sublaminates selected for design often use fixed ratios of ply orientations that have been previously tested and are well characterized. Because ratios of the ply orientations are fixed, designers approximate the properties of sublaminates using equivalent orthotropic properties. The maximal errors in bending stiffness and buckling loads due to use of equivalent properties are investigated. It is found that repeating half of the sublaminate on either side of the symmetry plane results in larger errors than repeating the entire sublaminate. For the former case, errors of up to about 20% are found even with four sublaminate repetitions, whereas for the latter case, errors were under 6%. Predicted buckling loads exhibit the largest errors when equivalent properties are used for plates with large aspect and load ratios.

Introduction

OPTIMIZATION of composite laminates may lead to designs that are tailored to the specific loading conditions used and that are sensitive to uncertainties in loads and material properties. Modeling and analysis of the different failure modes, such as microcracking and fatigue failure, are also problematic. Many failure modes have not been described fully or are difficult to model. For these reasons, in practice, laminates are usually tested before they are used in structural applications.

To reduce failure risks and design costs, industry usually employs sublaminates that have been well characterized through testing and detailed analysis. Sublaminates are chosen with fixed ratios of various ply orientations to tailor the stiffness and strength in different directions and to provide safety against off-design conditions. Sublaminates are also used in optimization models to reduce the number of design variables.[1]

Because the in-plane stiffness of a laminate with fixed ratios of ply thicknesses in different orientations is determined only by the total thickness, designers frequently simplify modeling by replacing the composite laminate in the model with an equivalent homogenized orthotropic material.[2,3] Using equivalent orthotropic material properties for sublaminates can result in errors in bending stiffness (D-matrix) terms. These errors are small when the number of sublaminates is large.[4] Often such errors are caught in the final design verification where detailed analyses using an exact stacking sequence of the optimized designs are performed, and, therefore, the use of equivalent properties has not been a safety risk. However, using equivalent properties for optimization can result in suboptimal designs when the errors are large. The objective of the paper is to investigate how large errors due to the use of equivalent properties can be for simple but representative problems of plate bending and buckling.

The paper begins with a stiffened panel design example that motivated the present study. In that example, using equivalent properties for the skin and stiffener laminates in sizing optimization led to large errors that ranged between 20 and 40%. Next, a study of a general laminate with repeating units of sublaminate approximates is presented. Analytical expressions are derived for the errors in bending stiffness and strains caused by using equivalent properties as function of sublaminate properties and number and type of sublaminate repetitions. The analytical models are then used to identify conditions that can lead to the largest errors. The maximal errors are also calculated for multiple repetitions of the sublaminate with two different stacking schemes. Finally, error maximization is performed for a 16-ply symmetric laminate under biaxial compression to identify sublaminate layups that caused the maximal buckling load errors under different aspect and loading ratios when approximated using equivalent properties.

Errors in Stiffened Panel Optimization due to Use of Equivalent Properties

Our interest in the effect of using equivalent or effective stiffness of laminates arose during preliminary design optimization of J-stiffened panels for a reusable launch vehicle liquid hydrogen tank developed by Rockwell, Inc. Equivalent properties are obtained by equating the actual in-plane stiffness matrix terms A_{ij} to those obtained for the in-plane stiffness constants for a single-layered equivalent material. The effective elastic properties for a sublaminate of thickness h are then given as

$$E_x = (1/h)\left[\left(A_{11}A_{12} - A_{12}^2\right)/A_{22}\right]$$

$$E_y = (1/h)\left[\left(A_{11}A_{22} - A_{12}^2\right)/A_{11}\right]$$

$$\nu_{xy} = A_{12}/A_{22}, \qquad G_{xy} = (1/h)A_{66} \tag{1}$$

The panel optimized was a cylindrical panel stiffened longitudinally (internal J-type stringers) and circumferentially (T-type rings, see Fig. 1) with 180-deg span, 192-in. radius, and 300-in. length. Figure 1 shows the stiffener geometry used with some associated terminology. The panel was designed to resist buckling failure under an axial compression load of 1741 lb/in. with a safety factor of 1.4. Details of the material properties and stacking sequence of the panel used for the analysis and design are shown in Tables 1 and 2. The panels were optimized using the PANDA2 program.[5] The optimization of the panels did not include the ring stiffeners or their spacing, but rings were included in the model to account for their stiffening effect.

A problem was discovered when we attempted to reconcile the differences in results obtained from two different models.[6] In one

Received 9 April 1999; revision received 22 August 2000; accepted for publication 22 August 2000.

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Table 1 Stiffner sizing and laminate layups of initial panel design

Description	Value[a]	Description	Value
Skin	$[\pm 65_{3T}]_s$	Ring spacing	31.447 in.
Stringer (str) spacing	5.0266 in.		
Str bottom flange width	1.500 in.	Ring bottom flange width	1.0 in.
Str top flange width	0.600 in.	Ring top flange width	2.0 in.
Str height	1.500 in.	Ring height	9.0 in.
Str bottom flange	$[0_{2F}]_s$	Bottom flange of ring	$[45_{1F}/0_{4T}/0_{1F}/0_{2T}]_s$
Str web	$[0_{1F}/0_{4T}/45_{1F}/0_{4T}/45_{1F}/0_{4T}/0_{1F}]_s$	Web of ring	$[45_{1F}/0_{2F}/45_{1F}]_T$
Str top flange	$[0_{1F}/0_{4T}/45_{1F}/0_{4T}/45_{1F}/0_{4T}/0_{1F}]_s$	Ring top flange	$[45_{1F}/0_{4T}/45_{1F}/0_{2T}]_s$

[a]Subscripts T and F denote preimpregnated tape lamina and woven fabric lamina, respectively.

Table 2 Material properties of tape and fabric plies used for panel design

	Lamina type	
Parameter	Tape	Fabric
Thickness of single ply, in.	0.005	0.0143
Elastic modulus in fiber direction E_1, Mpsi	21.5	11.6
Elastic modulus in transverse direction E_2, Mpsi	1.08	10.9
Poisson ratio ν_{12}	0.3	0.06
Shear modulus G_{12}, Mpsi	0.6	1.12
Maximum allowable tensile stress in fiber direction, ksi	222.3	113.7
Maximum allowable compressive stress in fiber direction, ksi	176.3	74.8
Maximum allowable tensile stress in transverse direction, ksi	7.2	103.9
Maximum allowable compressive stress in transverse direction, ksi	36.0	75.6
Maximum allowable shear stress, ksi	10.9	16.32

Table 3 Optimized panel designs obtained from models using equivalent properties and ply layup

Geometry variable	Optimum using with exact laminate layup	Optimum using equivalent properties
Stiffener spacing, in.	4.557	4.823
Stiffener height, in.	1.326	2.035
Bottom flange width, in.	1.165	1.324
Top flange width, in.	0.5806	0.683
Panel weight, lb	1484	1665

Table 4 Buckling load factors for stiffened panel using exact and equivalent properties[a]

		Buckling load factor of panels from analysis	
Panel	Panel weight, lb	Using equivalent properties	Using exact properties
Initial panel design	1496	0.99 (−20.8%)	1.25
Panel design optimized using equivalent properties	1665	1.40 (−41.2%)	2.38
Panel design optimized using exact properties	1484	0.93 (−33.6%)	1.40

[a]Layups and material properties given in Table 2.

model, the stiffened panel included the laminate stacking sequence details; in the other, segments of the stiffened panels were modeled as orthotropic materials using equivalent properties. The two panel models were optimized with stringer spacing, height, and flange widths as design variables. Table 3 presents the designs and total weight and shows that the optimum based on effective material properties was 11% heavier with larger stringers.

Table 4 shows the critical buckling load factor of the initial and optimum designs obtained using the two approaches. The initial design analyzed using ply layup details was critical in torsional buckling of the skin-stringer segment with a load factor of 1.253, whereas the analysis with equivalent stiffness predicted a critical local buckling

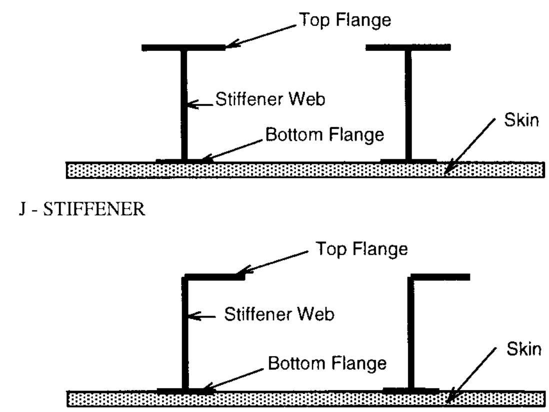

Fig. 1 Stiffener geometry and associated terminology.

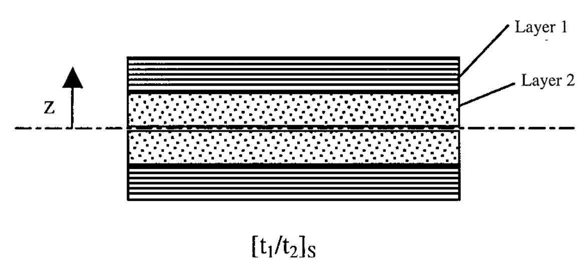

Fig. 2 Sublaminate with layers t_1 and t_2.

mode at load factor of 0.994. Table 4 also shows the buckling loads of both optimized designs using both analysis approaches. Optimization tends to take advantages of model weaknesses, and so the process amplified the modeling errors: Errors in the buckling load factor prediction increased from 21% for the initial design to 34 and 41% for the optimized panels.

Maximal Errors in Bending Stiffness and Strains for a Two-Ply Symmetric Laminate

Next we sought to find out how large the errors due to use of equivalent properties could get. In this section, a four-layer symmetric laminate is used to seek maximal errors in bending stiffness. A symmetric laminate (Fig. 2) with layup $[t_1/t_2]_s$ is chosen, where t_1 and t_2 are the thicknesses of layers 1 and 2, respectively. The laminate is a special case of laminates $[t_1/t_2]_{sn}$ and $[(t_1/t_2)_n]_s$ for $n = 1$. The bending stiffness of the laminate is calculated using exact and equivalent properties, and the volume fractions and stiffness ratio of the two plies are varied so as to find the maximum error.

Let $(\bar{Q}_{ij})_k$ be the in-plane stiffness of the kth layer, t_k the thickness of the kth layer, z^{k+1} and z^k the coordinates of the top and bottom faces of the kth layer, n_l the total number of layers in the laminate, and h the total thickness of the laminate. The laminate in-plane stiffness A_{ij} and bending stiffness D_{ij} are then expressed as follows:

$$A_{ij} = \sum_{k=1}^{n_l} (Q_{ij})_k t_k \tag{2}$$

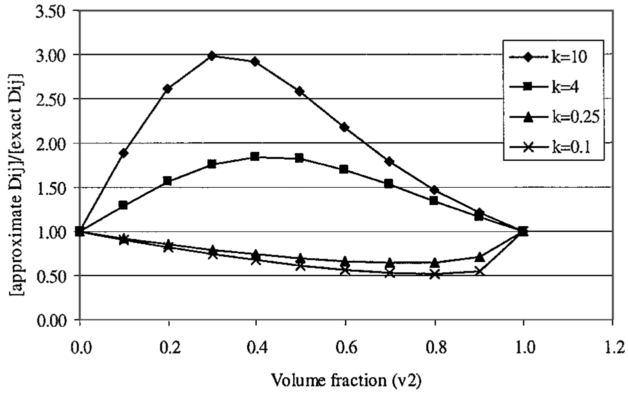

Fig. 3 Ratio of bending stiffness from equivalent and exact properties.

$$D_{ij} = \sum_{k=1}^{n_l} (\bar{Q}_{ij})_k \frac{\left(z_{k+1}^3 - z_k^3\right)}{3} \tag{3}$$

It is convenient to work with normalized stiffnesses obtained by dividing Eqs. (2) and (3) by h and $h^3/12$, respectively, to give A_{ij}^* and D_{ij}^*. For the four-layer symmetric laminate $[t_1/t_2]_s$, the normalized in-plane stiffness is

$$A_{ij}^* = \bar{Q}_{ij_1} v_1 + \bar{Q}_{ij_2} v_2 \tag{4}$$

where $v_1 = 2t_1/h$ and $v_2 = 2t_2/h$ are the volume fractions associated with the two orientations. The bending stiffness calculated from equivalent properties is

$$D_{ij_{\text{eq}}}^* = 12 D_{ij_{\text{eq}}} \big/ (h)^3 = A_{ij}^* \tag{5}$$

The exact bending stiffness is

$$D_{ij} = \tfrac{2}{3}\bar{Q}_{ij_2} t_2^3 + \tfrac{2}{3}\bar{Q}_{ij_1}\left[(t_1 + t_2)^3 - t_2^3\right] \tag{6}$$

The normalized bending stiffness expressed in terms of volume fraction of the second layer is

$$D_{ij}^* = \bar{Q}_{ij_2} v_2^3 + \bar{Q}_{ij_1}\left(1 - v_2^3\right) \tag{7}$$

The ratio of the bending stiffness from equivalent properties and exact calculation is

$$\frac{D_{ij_{\text{eq}}}^*}{D_{ij}^*} = \frac{A_{ij}^*}{D_{ij}^*} = \frac{\bar{Q}_{ij_2} v_2 + \bar{Q}_{ij_1}(1 - v_2)}{\bar{Q}_{ij_2} v_2^3 + \bar{Q}_{ij_1}\left(1 - v_2^3\right)} = \frac{k v_2 + (1 - v_2)}{k v_2^3 + \left(1 - v_2^3\right)} \tag{8}$$

where k is the ratio of an in-plane stiffness component of the two layers:

$$k = \bar{Q}_{ij_2} / \bar{Q}_{ij_1} \tag{9}$$

The ratio of bending stiffness from equivalent properties and exact calculation is calculated for different values of the in-plane stiffness ratio ($k = 10$, 4, 0.25, and 0.1). A value of $k = 10$ is typical of the ratio of A_{11} stiffness term for 0- and 90-deg plies of graphite–epoxy composites. A value of $k = 4$ corresponds to the ratio of A_{11} stiffness term for 0- and 45-deg plies. Values of $k = 0.25$ and 0.1 are reciprocals of $k = 4$ and 10, indicating that the outer layer is the stiffer material. Figure 3 shows the ratio of bending stiffness terms calculated using Eq. (8) for values of the volume fraction v_2 ranging between 0 and 1.

Table 5 lists the maximal values of the errors arising from using equivalent properties for a single sublaminate. The volume fractions corresponding to the maximal errors are also provided. The example laminates shown in the last column have volume fractions at or close to the maximal error value and will exhibit errors in D_{11} stiffness term similar to those shown in the second column.

As expected, larger maximal errors were found for $k = 10$ and 0.1, which indicates that errors are large when the difference in stiffness of plies is large. The error is more severe when the ply with higher stiffness is moved closer to the symmetry plane. Bending stiffness calculated using equivalent properties overestimates the contribution of the layer that is closer to the plane of symmetry. When the stiffer ply is away from the plane of symmetry ($k < 1$), the opposite happens, and equivalent properties underestimate the bending stiffness.

Table 5 Maximal errors in bending stiffness of a laminate calculated using equivalent properties

Stiffness ratio k	Maximal error in bending stiffness, %	Volume fraction v_2 at maximum error	Laminate layup that exhibits similar errors in D_{11} stiffness
10	198.0	0.3	$[90_2/0]_s$
4	84.6	0.4	$[\pm 45_3/0_4]_s$
0.25	−36.0	0.7	$[0_2/\pm 45_2]_s$
0.1	−49.1	0.8	$[0/90_4]_s$

Table 6 Error in bending curvatures due to using equivalent property for graphite-epoxy[a]

Laminate	Error in bending curvature K_x for pure cylindrical bending, %	Error in twist curvature K_{xy} for pure twist loading, %
$[90_2/0]_s$	−66.3	0.0
$[45_2/0_2]_s$	−34.9	+32.8
$[0_2/45_2]_s$	43.6	−41.0
$[45_2/0_2/90_2]_s$	−8.3	50.9

[a] $E_1 = 18.5$ Mpsi, $E_2 = 1.89$ Mpsi, $G_{12} = 0.93$ Mpsi, and $\nu = 0.3$.

Large errors are encountered in calculating bending strains when bending stiffnesses are calculated using equivalent properties. Table 6 shows the errors in curvatures calculated for pure cylindrical bending, κ_x, and pure twist, κ_{xy}, conditions for sample laminates. For more general loading conditions, the curvatures will depend on several bending stiffness D_{ij} terms that have compensating errors. The cylindrical bending and pure twist are, therefore, examples of possible worst-case scenarios.

The $[90_2/0]_s$ laminate shown in Table 5 is an example of a laminate that can give 200% error in D_{11}, that is, the approximate D_{11} is three times the exact one. The K_x curvature is proportional to the inverse of D_{11} term. Hence, the error is 66%, that is, the exact curvature is only one-third of the value obtained using equivalent properties. The negative sign indicates that bending stiffness calculated from equivalent properties underestimates the curvature (and, hence, bending stresses). The second and third examples in Table 6 show that the error in a strain component in a given direction depends on the position of the ply with orientation corresponding to that strain component. The bending strain K_x is underestimated when the 0-deg fiber is located at the center (near-symmetry plane), whereas the twist curvature is underestimated when the 45-deg layer is located close to the center. The bending curvature error is small for the last example because the 0-deg ply is located farther away from the center and the 90-deg ply that has the least stiffness in the x direction is located at the center.

Maximal Errors in Bending Stiffness of Laminates with Repeating Sublaminates

The large errors shown in Tables 5 and 6 are obtained from approximating the entire laminate with equivalent properties. In practice, equivalent properties are used for sublaminates that are repeated in thicker laminates. With many repeating sublaminates, the use of equivalent material properties will lead to smaller errors. This section investigates how fast the errors decay as the number of sublaminates increases.

The designer has two choices in the arrangement of the symmetric sublaminates (Fig. 4). One choice is to stack the n symmetric sublaminates on top of one another. For example, if the sublaminate has a layup of $[90/0]_s$, then the laminate layup with two sublaminates will be $([90/0]_s)_2$ or $[90/0_2/90]_2$. An alternative choice is to repeat the symmetric part of the sublaminate. In other words, use n stacks of the bottom and top halves of the sublaminate on either side of the symmetry plane of the resulting laminate. For example, three repetitions of a sublaminate with layup of $[90/0]_s$ will result in a laminate with layup $[(90/0)_3]_s$ or $[(90/0/90/0/90/0]_s$. In the

Table 7 Maximal error in D_{ij} terms for n full-sublaminate repetitions as in $[t_1/t_2]_{sn}$ laminate

Stiffness ratio k	Volume ratio of layer 2 v_2	Maximal error in D_{ij}, %			
		$n=1$	$n=2$	$n=3$	$n=4$
10.0	0.3	198.0	19.9	8.0	4.3
4.0	0.4	84.6	12.9	5.4	2.9
0.25	0.7	−36.0	−12.4	−5.9	−3.3
0.10	0.8	−49.1	−18.8	−9.3	−5.5

Table 8 Maximal error in D_{ij} terms for n half-sublaminate repetitions as in $[(t_1/t_2)n]_s$ laminate

Stiffness ratio k	Volume ratio of layer 2 v_2	Maximal error in D_{ij}, %			
		$n=1$	$n=2$	$n=3$	$n=4$
10.0	0.3	198.0	55.7	32.3	22.7
4.0	0.4	84.6	31.1	19.0	13.7
0.25	0.7	−36.0	−21.0	−14.8	−11.4
0.10	0.8	−49.1	−29.8	−21.5	−16.8

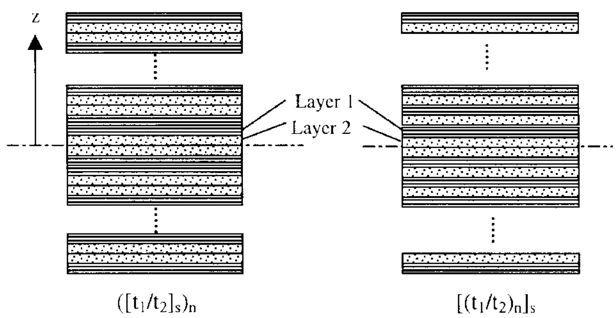

Fig. 4 Schemes for stacking sublaminates: full-sublaminate and half-sublaminate repetitions.

rest of the discussion, the first arrangement will be referred to as a full-sublaminate repetition, whereas the second will be referred to as a half-sublaminate repetition. The half-sublaminate repetition is often used because it permits more freedom to tailor thickness by dropping or adding more half-sublaminates.[1]

The expressions for the ratio of bending stiffness calculated using equivalent properties $D^*_{ij_{\rm eq}}$ and exact method D^*_{ij} are derived in the Appendix. The ratio of bending stiffness for full-sublaminate repetition is

$$\frac{D^*_{ij_{\rm eq}}}{D^*_{ij}} = n^2 \Big/ \left[\frac{D^*_{ij_{\rm sub}}}{A^*_{ij_{\rm sub}}} + (n-1)(n+1)\right] \tag{10}$$

where $A^*_{ij_{\rm sub}}$ and $D^*_{ij_{\rm sub}}$ are the in-plane and bending stiffnesses of the sublaminate. The bending stiffness ratio for laminates with half-sublaminate repetition is

$$\frac{D_{ij_{\rm eq}}}{D^*_{ij}} = n^2 \Big/ \left[\frac{D^*_{ij_{\rm sub}}}{A^*_{ij_{\rm sub}}} + \frac{1}{2}(n-1)(2n-1) - \frac{12(n-1)}{h^2 A^*_{ij_{\rm sub}}}\int_{-h/2}^{0} (\bar{Q}_{ij})_b z\,{\rm d}z\right] \tag{11}$$

The integral in the denominator is similar to the bending–extension coupling term, but it is evaluated only for one-half of the sublaminate.

The derived expressions are used to calculate the ratio of bending stiffness from the approximate and exact approach for the $[t_1/t_2]_n$ laminate for the two stacking schemes discussed. The ratios were calculated for values of volume fraction of layer 2, v_2, from 0.0 to 1.0 in increments of 0.1.

The maximal error in the bending stiffness for laminates with n full-sublaminate repetition is presented in Table 7. The value of the volume fraction resulting in maximal error is independent of the number of sublaminate repetitions, n. For n full-sublaminate repetitions, the error reduces rapidly as the number of sublaminates increase. The maximal error is strongly dependent on the ratio of the in-plane stiffness of the plies.

Table 8 shows the errors in bending stiffness for laminates with n half-sublaminate repetitions. For $n=1$, the laminate is the same as in Table 7 and, hence, exhibits the same maximal error. However, compared to full-sublaminate repetition, the error reduces more slowly when the number of sublaminates is increased. The maximal errors for $k=10$ are 55.7, 32.3, and 22.7%, respectively, for $n=2$, 3, and 4. The error is higher because the stiffer ply is located closer to the symmetry (reference) plane and thereby contributes less to the bending stiffness.

Let us consider the example of a $[90/0]_s$ sublaminate with two repetitions. Full-sublaminate repetition results in a laminate with layup of $[90/0_2/90]_s$, whereas the half-sublaminate repetition results in a layup of $[90/0/90/0]_s$. The D_{11} stiffness of the $[90/0/90/0]_s$ laminate is lower compared to that of $[90/0_2/90]_s$ laminate because the 0-deg ply is moved closer to the symmetry plane, which leads to larger errors when equivalent properties are used. The volume fraction at maximal error is identical for full-sublaminate and half-sublaminate repetitions.

Maximal Buckling Load Error

In the preceding section, maximal errors in bending stiffness were investigated for two-ply sublaminates. This section will investigate errors in buckling loads due to the use of equivalent properties for laminates made from 0-, ±45-, and 90-deg plies. To estimate the maximal errors in buckling loads, laminates were optimized for maximal error, a process known as antioptimization.[7]

The error in buckling load λ_c due to the use of equivalent properties was calculated for a simply supported rectangular plate in biaxial compression. The buckling load is

$$\lambda_c = \frac{\pi^2\left[D_{11}(m/a)^4 + 2(D_{12}+2D_{66})(m/a)^2(n/b)^2 + D_{22}(n/b)^4\right]}{\left[N_x(m/a)^2 + N_y(n/b)^2\right]} \tag{12}$$

where N_x and N_y are the loads in the axial and transverse directions, a and b are the dimensions of the plate, and m and n are the number of half-waves in the axial and transverse directions.

The percentage relative error in buckling loads due to equivalent laminate properties is defined as

$$\%{\rm error} = 100\frac{\left(\lambda_c^{\rm approx} - \lambda_c^{\rm exact}\right)}{\lambda_c^{\rm exact}} \tag{13}$$

where $\lambda_c^{\rm approx}$ is the buckling load factor calculated using equivalent laminate properties and $\lambda_c^{\rm exact}$ is the buckling load factor calculated using exact ply representation.

The antioptimization of the laminate layups was performed using a genetic algorithm to maximize the absolute relative error in buckling load [Eq. (13)] for 16-ply symmetric and balanced laminates. The design variables of the genetic algorithm were ply orientations that were restricted to ±45, 0, and 90 deg. A genetic algorithm uses operators that mimic biological evolutionary processes, such as mutation and crossover, to obtain improved designs. The operators work on a population of designs, each represented by a chromosome. The selection of the chromosome for reproduction depends on the fitness value assigned to the chromosome. The fitness value here is the magnitude of the error. The genes of the chromosome (which determine the characteristics of the design) are the encoded optimization variables. For laminate stacking sequence design, the discrete ply orientation angles are coded using integer variables.

To ensure that all laminates obtained are balanced, the basic unit of coding represents a stack of two plies: 0_2, 90_2 or ±45. With the two-ply stack coding, the 16-ply symmetric laminate is represented using a chromosome with four genes. The errors were obtained for a laminate made of graphite–epoxy plies with properties $E_1 = 18.5$ Mpsi, $E_2 = 1.89$ Mpsi, $G_{12} = 0.93$ Mpsi, and $\nu_{12} = 0.3$.

The stacking sequence antioptimization is performed for plates with varying aspect ratios (a/b) and load ratios (N_y/N_x). The aspect ratio and loading ratio were varied from $\frac{1}{16}$ to 16, with steps of factors of 2.0 giving 81 cases in all. The maximal value of the errors in buckling load factor and the resulting laminate for a range of aspect ratios and load ratios are shown in Table 9, with laminates having maximal errors given in Table 10.

For the 16-ply laminate optimized, the largest maximal error (180%) occurs when there is a combination of high aspect ratio and high load ratio. The smallest maximal error (22%) is obtained for a plate with unit aspect ratio and load ratio. Of all of the possible 81 sublaminates, only a small set of 10 (codes A–E and L–P) was found in the antioptimization, for the case of one sublaminate through the thickness. The large errors shown in Table 9 are for the extreme case where the laminate has a single sublaminate.

Table 9 Maximal buckling load errors (%) of a single sublaminate[a]

Load	Aspect ratio a/b								
N_x/N_y	0.0625	0.125	0.250	0.500	1.000	2.000	4.000	8.000	16.00
0.0625	58 (N)	58 (N)	59 (N)	65 (N)	51 (A)	94 (E)	154 (B)	175 (B)	180 (B)
0.125	67 (N)	65 (N)	66 (N)	81 (O)	51 (A)	94 (E)	154 (B)	175 (B)	180 (B)
0.250	101 (M)	101 (M)	104 (M)	90 (M)	41 (A)	94 (E)	154 (B)	175 (B)	180 (B)
0.500	162 (M)	159 (M)	149 (M)	94 (L)	24 (A)	94 (E)	154 (B)	175 (B)	180 (B)
1.000	179 (O)	175 (O)	154 (O)	94 (L)	22 (N)	94 (E)	154 (B)	175 (B)	179 (B)
2.000	180 (O)	175 (O)	154 (O)	94 (L)	24 (P)	94 (E)	149 (D)	159 (D)	162 (D)
4.000	180 (O)	175 (O)	154 (O)	94 (L)	41 (P)	90 (D)	104 (D)	101 (D)	101 (D)
8.000	180 (O)	175 (O)	154 (O)	94 (L)	51 (P)	81 (B)	65 (C)	65 (C)	67 (C)
16.00	180 (O)	175 (O)	154 (O)	94 (L)	51 (P)	65 (C)	59 (C)	58 (C)	58 (C)

[a]Coding (A–P) refers to the stacking sequences in Table 10.

Table 10 Stacking sequence of sublaminates optimized for maximal error in buckling load for one sublaminate in total laminate

Code	Sublaminate layup	Code	Sublaminate layup
A	$[0_2/0_2/0_2/\pm 45]_s$	I	$[\pm 45/90_2/90_2/90_2]_s$
B	$[0_2/0_2/0_2/90_2]_s$	J	$[90_2/0_2/0_2/0_2]_s$
C	$[0_2/0_2/\pm 45/\pm 45]_s$	K	$[90_2/0_2/\pm 45/\pm 45]_s$
D	$[0_2/0_2/\pm 45/90_2]_s$	L	$[90_2/90_2/0_2/0_2]_s$
E	$[0_2/0_2/90_2/90_2]_s$	M	$[90_2/90_2/\pm 45/0_2]_s$
F	$[0_2/90_2/\pm 45/\pm 45]_s$	N	$[90_2/90_2/\pm 45/\pm 45]_s$
G	$[0_2/90_2/90_2/90_2]_s$	O	$[90_2/90_2/90_2/0_2]_s$
H	$[\pm 45/0_2/0_2/0_2]_s$	P	$[90_2/90_2/90_2/\pm 45]_s$

A select number of cases from Table 9 were chosen for further investigation of the effect of the number of sublaminates on buckling load error. This is shown in Table 11 for a laminate with half-sublaminate repetition and in Table 12 for full-sublaminate repetition.

The errors decay more slowly for half-sublaminate repetition. The decay in errors is consistent with those observed for the bending stiffness errors (Tables 7 and 8).

Large errors in buckling loads are observed for plates with large aspect and load ratios (Table 9). Smaller buckling load errors (21.5%) were found for the $[90_4/\pm 45_2]_s$ laminate (Tables 11 and 12) obtained from antioptimization of a plate with unit aspect ratio and load ratio. At unit aspect ratio, the dominant term in the buckling equation [Eq. (13)] is the D_{66} term. The maximal error occurs when the stiffer layer (±45-deg layers) is closer to the symmetry plane. The reduced errors at unit aspect and load ratios are also due to the compensating errors in the different bending stiffness terms [Eq. (13)]. For plates with high (or low) aspect and load ratios, the buckling loads are more dependent on individual bending stiffness terms D_{11} and D_{22}. For the $[90_4/\pm 45/0_2]_s$ laminate of Tables 11 and 12, equivalent properties underestimate the D_{11} stiffness term and overestimate the D_{22} stiffness term, which results in large errors in buckling loads.

Summary

We have sought conditions when using equivalent or effective material properties for laminates can lead to large errors in bending stiffness. The four-ply sublaminate example showed that the bending stiffness terms calculated using equivalent properties could have large errors (198%). The maximal errors are a function of the ratio of in-plane stiffness of the plies and their respective volume fractions. Large errors occur when the equivalent properties overestimate the contribution of plies that are closer to the center or symmetry plane. When plies with lower stiffness are moved to the center of the laminate, the error is smaller, and the approximation is conservative. Bending strains obtained from models using equivalent properties can result in large errors.

For multiple sublaminates, the severity of the errors is reduced when the entire sublaminate is repeated. The error decays more slowly for laminates with half-sublaminate repetition, resulting in large errors (22.7%) even for four repetitions of the sublaminate.

Antioptimization of laminates for maximal buckling load errors showed that errors in using equivalent properties could result in underestimation or overestimation. Large buckling load errors (180%) were encountered for plates with high aspect ratio and/or load ratio. The high aspect ratio and loading ratio are typical of stiffened panel segments, such as local skin portion between stiffeners or the stiffener webs and flanges.

The error decayed rapidly as the number of sublaminates increased, with maximal errors of 20 and 7%, respectively, for two

Table 11 Effect of number of sublaminates in half sublaminate repetition on error in buckling load calculated using equivalent properties

Case			Worst-case	% Error in buckling load for multiple sublaminates				
no.	N_x/N_y	a/b	sublaminate	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
1	0.5	1/16	$[(90_4/\pm 45/0_2)_n]_s$	162.1	45.85	26.09	18.20	13.90
2	0.5	0.5	$[(90_4/0_4)_n]_s$	94.22	32.02	19.29	13.40	10.75
3	1.0	1.0	$[(90_4/\pm 45_2)_n]_s$	21.50	9.71	6.27	4.63	3.67

Table 12 Effect of number of sublaminates in full sublaminate repetition on error in buckling load calculated using equivalent properties

Case			Worst-case	% Error in buckling load for multiple sublaminates				
no.	N_x/N_y	a/b	sublaminate	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
1	0.5	1/16	$[90_4/\pm 45/0_2]_{sn}$	162.1	16.54	6.57	3.55	2.24
2	0.5	0.5	$[90_4/0_4]_{sn}$	94.22	13.80	5.70	3.21	1.98
3	1.0	1.0	$[90_4/\pm 45_2]_{sn}$	21.50	4.63	2.00	1.11	0.71

and three full-sublaminate repetitions. The errors decay more slowly for laminates with half-sublaminate repetitions. The compensating errors in the different bending stiffness terms result in smaller errors in the buckling loads.

Using equivalent properties in optimization models can lead to suboptimal designs because optimization programs are notorious for exploiting model errors. To ensure that the errors are smaller than 20%, the designer should avoid using equivalent properties 1) for approximating entire laminates, 2) for laminates with fewer than three full-sublaminate repetitions, 3) for laminates with fewer than five half-sublaminate repetitions, 4) for bending stress calculations, and 5) for calculating buckling loads of plates with high aspect ratio and load ratios.

These conclusions are based on examination of errors in buckling loads, bending stiffness, and strains of laminated composites under simple biaxial loads for a small range of aspect and load ratios. The reader is cautioned not to extend these to more general classes of composite laminates (such as asymmetric laminates) or to different loading conditions.

Appendix: Bending Stiffness of a Laminate with Multiple Sublaminates

Analytical expressions for ratio of the bending stiffness obtained from equivalent properties and an exact approach are derived for symmetric laminates with n sublaminate repetitions. The expressions are derived for the two different arrangements of the sublaminate, namely, full-sublaminate repetitions and half-sublaminate repetitions.

Laminate with Full-Sublaminate Repetitions

Let us consider a laminate with n symmetric sublaminates stacked on top of one another, as in $[t_1/t_2]_{sn}$ laminate. Let $\bar{Q}_{ij}$ be the in-plane stiffness of the material. The laminate in-plane stiffness A_{ij} and bending stiffness D_{ij} obtained from lamination theory are expressed as

$$A_{ij} = \int_{-nh/2}^{nh/2} (\bar{Q}_{ij})\,\mathrm{d}z = \sum_{k=1}^{n_l} (Q_{ij})_k t_k \tag{A1}$$

$$D_{ij} = \int_{-nh/2}^{nh/2} (\bar{Q}_{ij})\,z^2\,\mathrm{d}z = \sum_{k=1}^{n_l} \left(\bar{Q}_{ij}\right)_k \frac{\left(z_{k+1}^3 - z_k^3\right)}{3} \tag{A2}$$

where n is the total number of sublaminates, h is the thickness of the sublaminate, t_k is the thickness of the kth layer (ply), z^{k+1} and z^k are the distances to the top and bottom faces of the kth layer from the reference surface, and n_l is the total number of layers (plies) in the laminate (Fig. 3). The in-plane and bending stiffness are normalized by the quantities nh and $(nh)^3/12$ to give A^*_{ij} and D^*_{ij}.

If we use the equivalent material properties to calculate bending stiffness, the normalized equivalent bending stiffness is given by the expression

$$D^*_{ij_{\mathrm{eq}}} = \frac{12 D_{ij_{\mathrm{eq}}}}{(nh)^3} = A^*_{ij} \tag{A3}$$

The exact bending stiffness of the laminate is given by the expression

$$D_{ij} = \int_{-nh/2}^{nh/2} (\bar{Q}_{ij}) z^2\,\mathrm{d}z = \int_{-nh/2}^{-(n-2)h/2} (\bar{Q}_{ij}) z^2\,\mathrm{d}z + \cdots + \int_{-h/2}^{h/2} (\bar{Q}_{ij}) z^2\,\mathrm{d}z + \cdots + \int_{(n-2)h/2}^{2h/2} (\bar{Q}_{ij}) z^2\,\mathrm{d}z \tag{A4}$$

Rewriting the integrals for each sublaminate with the same integration limits results in

$$D_{ij} = \sum_{p=1}^{n} \int_{-nh/2+(p-1)h}^{-nh/2+ph} (\bar{Q}_{ij}) z^2\,\mathrm{d}z = \sum_{p=1}^{n} \int_{-h/2}^{h/2} (\bar{Q}_{ij}) \left[z - (n+1)\frac{h}{2} + ph \right]^2 \mathrm{d}z \tag{A5}$$

Expanding and regrouping the integrals give

$$D_{ij} = \left(\sum_{p=1}^{n} 1 \right) \int_{-h/2}^{h/2} (\bar{Q}_{ij})(z)^2\,\mathrm{d}z + \left\{ \sum_{p=1}^{n} \left[(n+1)\frac{h}{2} - ph \right]^2 \right\} \left(\frac{h}{2} \right)^2 \int_{-h/2}^{h/2} (\bar{Q}_{ij})\,\mathrm{d}z + 2 \left\{ \sum_{p=1}^{n} \left[(n+1)\frac{h}{2} - ph \right] \right\} \left(\frac{h}{2} \right)^2 \int_{-h/2}^{h/2} (\bar{Q}_{ij}) z\,\mathrm{d}z \tag{A6}$$

Recognizing the integrals in Eq. (A6) as the bending, in-plane, and bending–extension coupling stiffness terms of the sublaminate, respectively, denoted as $D_{ij_{\mathrm{sub}}}$, $A_{ij_{\mathrm{sub}}}$, and $B_{ij_{\mathrm{sub}}}$, we can rewrite Eq. (A6) as

$$D_{ij} = \left(\sum_{p=1}^{n} 1 \right) D_{ij_{\mathrm{sub}}} + \left\{ \sum_{p=1}^{n} \left[(n+1)\frac{h}{2} - ph \right]^2 \right\} \left(\frac{h}{2} \right)^2 A_{ij_{\mathrm{sub}}} + 2 \left\{ \sum_{p=1}^{n} \left[(n+1)\frac{h}{2} - ph \right] \right\} \left(\frac{h}{2} \right)^2 B_{ij_{\mathrm{sub}}} \tag{A7}$$

For symmetric sublaminates, $B_{ij_{\mathrm{sub}}}$ is zero. Simplifying the summations in Eq. (A7), we obtain

$$D_{ij} = n D_{ij_{\mathrm{sub}}} + [(n-1)(n)(n+1)/3](h/2)^2 A_{ij_{\mathrm{sub}}} \tag{A8}$$

Normalizing D_{ij} in Eq. (A8) by $(nh)^3/12$ gives

$$D^*_{ij} = (1/n^2) D^*_{ij_{\mathrm{sub}}} + [(n-1)(n+1)/n^2] A^*_{ij_{\mathrm{sub}}} \tag{A9}$$

The ratio of bending stiffness of laminate calculated using equivalent properties and exact approach is then given by the expression

$$\frac{D^*_{ij_{\mathrm{eq}}}}{D^*_{ij}} = n^2 \Bigg/ \left[\frac{D^*_{ij_{\mathrm{sub}}}}{A^*_{ij_{\mathrm{sub}}}} + (n-1)(n+1) \right] \tag{A10}$$

Laminate with Half-Sublaminate Repetitions

Let us next consider a laminate with half-sublaminate repetitions. The laminate with half-sublaminate repetitions has the same in-plane stiffness properties as the laminate with full-sublaminate repetitions, but has different bending properties due to the difference in the stacking sequence. Because the in-plane stiffness does not change, the equivalent bending stiffness is same as before [Eq. (A3)].

Let $(\bar{Q}_{ij})_b$ and $(\bar{Q}_{ij})_t$ be the in-plane stiffness of the plies in the bottom and top half of the sublaminate. The bending stiffness of the laminate can be expressed as

$$D_{ij} = \int_{-nh/2}^{nh/2} (\bar{Q}_{ij}) z^2\,\mathrm{d}z = \int_{-nh/2}^{-(n-1)h/2} (\bar{Q}_{ij})_b z^2\,\mathrm{d}z + \cdots + \int_{-h/2}^{0} (\bar{Q}_{ij})_b z^2\,\mathrm{d}z + \int_{0}^{h/2} (\bar{Q}_{ij})_t z^2\,\mathrm{d}z + \cdots + \int_{(n-1)h/2}^{nh/2} (\bar{Q}_{ij})_t z^2\,\mathrm{d}z \tag{A11}$$

Rewriting all integrals of the bottom and top sublaminates using the same limits and writing the preceding expression as a summation series give

$$D_{ij} = \sum_{p=1}^{n} \int_{-h/2}^{-0} (\bar{Q}_{ij})_b \left[z - \frac{(p-1)h}{2} \right]^2 \mathrm{d}z + \sum_{p=1}^{n} \int_{0}^{h/2} (\bar{Q}_{ij})_t \left[z + \frac{(p-1)h}{2} \right]^2 \mathrm{d}z \tag{A12}$$

Expanding the integrals and regrouping similar terms gives

$$D_{ij} = \left(\sum_{p=1}^{n} 1\right)\left[\int_{-h/2}^{0} (\bar{Q}_{ij})_b z^2\,\mathrm{d}z + \int_{0}^{h/2} (\bar{Q}_{ij})_t z^2\,\mathrm{d}z\right] + \left(\frac{h}{2}\right)^2\left[\sum_{p=1}^{n}(p-1)^2\right]\left[\int_{-h/2}^{0} (\bar{Q}_{ij})_b\,\mathrm{d}z + \int_{0}^{h/2} (\bar{Q}_{ij})_t\,\mathrm{d}z\right] - h\left(\sum_{p=1}^{n}(p-1)\right)\left[\int_{-h/2}^{0} (\bar{Q}_{ij})_b z\,\mathrm{d}z - \int_{0}^{h/2} (\bar{Q}_{ij})_t z\,\mathrm{d}z\right] \tag{A13}$$

Recognizing the first two terms of Eq. (A13) as the bending and in-plane stiffness of the sublaminate, we can write

$$D_{ij} = nD_{ij_{\mathrm{sub}}} + \left(\frac{h}{2}\right)^2 \frac{1}{6} n(n-1)(2n-1)A_{ij_{\mathrm{sub}}} - \frac{h}{2}n(n-1)\left[\int_{-h/2}^{-0} (\bar{Q}_{ij})_b z\,\mathrm{d}z - \int_{-h/2}^{0} (\bar{Q}_{ij})_t z\,\mathrm{d}z\right] \tag{A14}$$

For symmetric sublaminates, the last term in Eq. (A14) can be further simplified because the two integrals are equal in magnitude but differ in sign. Normalizing the bending stiffness by $(nh)^3/12$ results in

$$D^*_{ij} = \frac{1}{n^2} D^*_{ij_{\mathrm{sub}}} + \frac{1}{2}\frac{(n-1)(2n-1)}{n^2} A^*_{ij_{\mathrm{sub}}} - \frac{12(n-1)}{n^2 h^2}\int_{-h/2}^{0} (\bar{Q}_{ij})_b z\,\mathrm{d}z \tag{A15}$$

The ratio of the bending stiffness obtained from equivalent properties and exact method for laminate $[(t_1/t_2)]_s$ with n half-sublaminate repetitions is

$$\frac{D_{ij_{\mathrm{eq}}}}{D^*_{ij}} = n_2 \Bigg/ \left[\frac{D^*_{ij_{\mathrm{sub}}}}{A^*_{ij_{\mathrm{sub}}}} + \frac{1}{2}(n-1)(2n-1) - \frac{12(n-1)}{h^2 A^*_{ij_{\mathrm{sub}}}}\int_{-h/2}^{0} (\bar{Q}_{ij})_b z\,\mathrm{d}z\right] \tag{A16}$$

The laminate with n repetitions of the half-sublaminate requires the evaluation of the following integral [last term in Eq. (A16)], in addition to quantities $A^*_{ij_{\mathrm{sub}}}$ and $D^*_{ij_{\mathrm{sub}}}$:

$$\int_{-h/2}^{0} (\bar{Q}_{ij})_b z\,\mathrm{d}z \tag{A17}$$

Evaluating this integral for the two-ply sublaminate (Fig. 2) results in the expression

$$\int_{-h/2}^{0} (\bar{Q}_{ij})_b z\,\mathrm{d}z = \bar{Q}_{ij_2}\left(\frac{t_2^2}{2}\right) + \bar{Q}_{ij_1}\left[\frac{(t_1+t_2)^2}{2} - \frac{t_2^2}{2}\right] \tag{A18}$$

$$\int_{-h/2}^{0} (\bar{Q}_{ij})_b z\,\mathrm{d}z = \frac{h^2}{2}\bar{Q}_{ij_2}\left[v_2^2 + k\left(1-v_2^2\right)\right] \tag{A19}$$

$$\int_{-h/2}^{0} (\bar{Q}_{ij})_b z\,\mathrm{d}z \Big/ h^2 A^*_{ij_{\mathrm{sub}}} = \frac{1}{2}\left[\frac{kv_2^2 + \left(1-v_2^2\right)}{kv_2 + (1-v_2)}\right] \tag{A20}$$

Acknowledgments

This work was supported by the Metals and Thermal Structures Branch, NASA Langley Research Center, under Grants NAG-1-1669 and NAG-1-1808. The authors express their thanks to David Bushnell of Lockheed Martin Company for his support with using the PANDA2 program.

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